The Degree of Generality of Inflation in FRW Models with Massive Scalar Field and Hydrodynamical Matter

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Abstract

Friedmann-Robertson-Walker cosmological models with a massive scalar field are studied in the presence of hydrodynamical matter in the form of a perfect fluid. The ratio of the number of solutions without inflation to the total number of solutions is evaluated, depending on the fluid density. It is shown that in a closed model this ratio can reach 60 %, by contrast to ~ 30 % in models without fluid.

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1   Introduction

In the recent two decades the dynamics of an isotropic Universe filled with a massive scalar field attracted a great attention [1]. From the physical point of view, the most interesting regime is the inflationary one. During inflation the system "forgets" its initial conditions and other characteristics of the pre-inflationary era, such as the possible presence of other types of matter in addition to the scalar field, spatial curvature, etc., due to a rapid growth of the scale factor. This feature enables us to use such a simple model for describing the physics of the early Universe from the instant when the inflationary regime was established.

On the other hand, the problem of pre-inflationary era and initial condition for inflation is for the same reason very difficult because nowadays we have no physical "probe" which might give us information about that time. In such a situation we may hope to extract some information from mathematical studies of the corresponding dynamical system. One of the most important problems is to describe the set of initial conditions which led to the inflationary regime. It can also clarify whether this regime is natural for this dynamical system or it requires some kind of fine tuning of the initial data.

For this question to make sense, it is necessary to specify a measure on the space of initial conditions. It is common to use the hypersurface with the energy density equal to the Planckean one (called the Planck boundary) as the initial-condition space. A common angular measure on the Planck boundary will be described below. As was pointed out in Ref. [3] (see also a detailed discussion of possible choices of the measure in the cited paper), this choice is based on the physical considerations of inapplicability of classical gravity beyond the Planck boundary and of the absence of any information from this region. When the measure is specified, the inflation generality problem can be studied quantitatively.

A solution can depend on the physical condition in the epoch followed by inflation. In Refs. [2, 4, 5] it was found that the distribution of initial data leading to inflation strongly depends on the sign of the spatial curvature. If it is negative or zero, the scale factor of the Universe cannot pass through extremum points (see below). In this case all the trajectories in the configuration space (a, φ), where a is the scale factor and φ is the scalar field, starting from a sufficiently large value φ_0, reach a slow-roll regime and experience inflation. If we start from the Planck energy, a measure of non-inflating trajectories is about m/m_P where m is the mass of the scalar field and m_P is the Planck mass. From observational reasons, this ratio is about 10^{-5} [6, 7] so almost all trajectories lead to the inflationary regime. But positive spatial curvature allows a trajectory to have a point of maximal expansion which results in increasing the measure of non-inflating trajectories to ~ 0.3 [4, 5].

Another important characteristic of the pre-inflationary era which is also "forgotten" during inflation is the possible presence of a hydrodynamical matter in addition to the scalar field. Decreasing even more rapidly than the curvature with increasing scale factor a, the energy density of hydrodynamical matter could not affect the slow-rolling conditions and so has a tiny effect on the dynamics if the spatial curvature is nonpositive. But the conditions for extrema of a can change significantly in a closed model, so the latter with a scalar field and hydrodynamical matter requires special analysis.

The structure of this paper is as follows: in Sec. 2 we consider the dynamical system corresponding to an isotropic Universe with a massive scalar field and without any other type of matter. We also show how to use the configuration space (a, φ) for illustrating the generality of inflation problem. In Sec. 3 this method is applied to a closed isotropic Universe with scalar field and hydrodynamical matter in the form of perfect fluid. In Sec. 4 we briefly discuss the generality of the results obtained in Sec. 3.

2   Basic equations and dimensionless variables

We consider a cosmological model with the action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{m_P^2}{16\pi} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} m^2 \varphi^2 \right\}. \qquad (2.1)$$

For a closed Friedmann model with the metric

$$ds^2 = dt^2 - a^2(t) d^2 \Omega^{(3)}, \qquad (2.2)$$

where $a(t)$ is the scale factor, $d^2\Omega^{(3)}$ is the metric on a unit 3-sphere and φ is a homogeneous scalar field, we can get the following equations of motion

$$\frac{m_P^2}{16\pi} \left(\ddot{a} + \frac{\dot{a}^2}{2a} + \frac{1}{2a} \right) + \frac{a\dot{\varphi}^2}{8} - \frac{m^2\varphi^2 a}{8} = 0 \qquad (2.3)$$

$$\ddot{\varphi} + \frac{3\dot{\varphi}\dot{a}}{a} + m^2\varphi = 0. \qquad (2.4)$$

Besides, we can write down the first integral of motion for our system

$$-\frac{3}{8\pi} m_P^2 (\dot{a}^2 + 1) + \frac{a^2}{2} \left(\dot{\varphi}^2 + m^2\varphi^2 \right) = 0. \qquad (2.5)$$

It is easily seen from (2.5) that the points of maximal expansion and contraction, i.e. the points where $\dot{a} = 0$ can exist only in a region where

$$\varphi^2 \le \frac{3}{4\pi} \frac{m_P^2}{m^2 a^2}, \qquad (2.6)$$

which represents the field in the half-plane $0 \le a < +\infty$, $-\infty < \varphi < +\infty$ bounded by the hyperbolic curves

$$\varphi \le \sqrt{\frac{3}{4\pi}} \frac{m_P}{ma} \quad \textbf{and} \quad \varphi \ge -\sqrt{\frac{3}{4\pi}} \frac{m_P}{ma}$$

(see Fig. 1). Sometimes the region determined by the inequalities (2.6) is called Euclidean or "classically forbidden". One can argue about the validity of such a definition (for details see [8]), but we shall use it for convenience. Now we would like to distinguish between the maximal contraction points where $\dot{a} = 0, \ddot{a} > 0$ and those of maximal expansion where $\dot{a} = 0, \ddot{a} < 0$. Let us put $\dot{a} = 0$, in this case one can express $\dot{\varphi}^2$ from (2.5) as

$$\dot{\varphi}^2 = \frac{3}{4\pi} \frac{m_P^2}{a^2} - m^2\varphi^2. \qquad (2.7)$$

Substituting (2.7) and $\dot{a} = 0$ into Eq. (2.3), we have

$$\ddot{a} = \frac{4\pi m^2 \varphi^2 a}{m_P^2} - \frac{2}{a}. \qquad (2.8)$$

3

From (2.8) one can easily see that the possible points of maximal expansion are localized inside the region

$$\varphi^2 \leq \frac{1}{2\pi} \frac{m_P^2}{m^2 a^2}, \tag{2.9}$$

while those of maximal contraction (bounces) lie outside the region (2.9) being at the same time inside the Euclidean region (2.6) (see Fig. 1) [9].

It is convenient to employ the dimensionless quantities

$$x = \frac{\varphi}{m_P}, \qquad y = \frac{\dot{\varphi}}{m m_P}, \qquad z = \frac{\dot{a}}{ma}.$$

We will study the dynamics of the Universe starting from the Planck boundary

$$\frac{m^2 \varphi^2}{2} + \frac{\dot{\varphi}^2}{2} = m_P^4 \tag{2.10}$$

We also introduce the convenient angular parametrization of the Planck boundary

$$\frac{m^2 \varphi^2}{2} = m_P^4 \cos^2 \phi, \qquad \frac{\dot{\varphi}^2}{2} = m_P^4 \sin^2 \phi. \tag{2.11}$$

This variable ϕ along with the variable z determine the initial point on the Planck boundary completely. z can vary in a compact region from 0 to $z_{\max} = \sqrt{8\pi/3}(m_P/m)$; the corresponding initial values of the scale factor a vary from $a_{\min} = \sqrt{3/(8\pi m_P^2)}$ to $+\infty$.

The measure we have used is the area S of the initial conditions on the (z, ϕ) plane over the total area S_{Pl} of the Planck boundary.

All definitions and properties of these variables remain unchanged in the presence of ordinary matter [except for adding the matter energy in the left-hand side of (2.10)], which will be studied in the next section. And now we briefly recall the situation without any matter in addition to the massive scalar field. Though all plots will be in the (z, ϕ)-plane, it is useful to keep in mind the (a, φ) configuration space.

As we know from [9], the fate of a trajectory starting from $\dot{a} = 0$ can be deduced from the coordinates of the initial point: a trajectory with $z = 0$ and an initial point lying between the two hyperbolae of Fig. 1 will expand, while an initial point lying below the separating curve leads to contraction instead of the inflationary regime. It is clear from (2.6), (2.9) that the scalar field value on the Euclidean boundary is $\sqrt{2/3}$ times the value of the scalar field on the separating curve for a fixed value of the scale factor. In the angular parametrization (2.11), the value of φ_0 lying on the separating curve and separating these two regimes corresponds to $\phi = \arccos(\sqrt{2/3}) \sim 0.61$. An initial, sufficiently large positive Hubble constant gives a the universe a chance to pass a "dangerous" region of possible maximal expansion points (to the left from the separating curve) and to reach the inflationry regime. The resulting distribution of initial points on the plane (z, ϕ) which do not lead to inflation is as shown in Fig. 2(a). A total measure of such trajectories is about 30% of the plane (z, ϕ).

4

3 A model with scalar field and perfect fluid

Now we add a perfect fluid with the equation of state $P = \gamma E$. The parameter γ can in principle vary in the range $-1 \leq \gamma \leq 1$. In this paper the case $-1/3 < \gamma \leq 1$ will be considered. This case contains all known kinds of matter in the form of a perfect fluid except the cosmological constant. Three cases of particular physical interest are $\gamma = 0$ (dust), $\gamma = 1/3$ (ultrarelativistic matter) and $\gamma = 1$ (massless scalar field).

The equation of motion are now

$$\frac{m_P^2}{16\pi}\left(\ddot{a} + \frac{\dot{a}^2}{2a} + \frac{1}{2a}\right) + \frac{a\dot{\varphi}^2}{8} - \frac{m^2\varphi^2 a}{8} - \frac{Q}{12a^{p+1}}(1-p) = 0 \qquad (3.1)$$

$$\ddot{\varphi} + \frac{3\dot{\varphi}\dot{a}}{a} + m^2\varphi = 0. \qquad (3.2)$$

with the constraint

$$-\frac{3}{8\pi}m_P^2(\dot{a}^2 + 1) + \frac{a^2}{2}\left(\dot{\varphi}^2 + m^2\varphi^2\right) + \frac{Q}{a^p} = 0. \qquad (3.3)$$

Here $p = 1 + 3\gamma$, Q is a constant from the equation of motion for matter which can be integrated in the form

$$Ea^{p+2} = Q = const. \qquad (3.4)$$

Before presenting the results of a numerical integration of (3.1)–(3.3) let us make some qualitative statements.

The Euclidean region is now bounded from large values of the field φ and small values of the scale factor a (see Fig. 1). The upper point of the Euclidean boundary $\varphi = \varphi_{\max}$ corresponds to $a^p = 4\pi(p+2)Q/(3m_P^2)$ and the fact that there is no bounce for bigger values of the scalar field is related to a transition from chaotic to regular types of dynamics, described by (3.1)–(3.3) as we have shown in [10].

But for our present purposes it is important that the Euclidean region disappears at

$$a^p = \frac{8\pi}{3m_P^2}Q. \qquad (3.5)$$

We also need the equation for $\ddot{a}$ at the points of bounce:

$$\ddot{a} = -\frac{2}{a} + \frac{4\pi}{m_P^2}m^2 a\varphi^2 + \frac{4\pi}{3m_P^2}\frac{Q}{a^{p+1}}(4-p). \qquad (3.6)$$

Thus for $p = 4$ (a massless scalar field) the separating curve is the same as it was for the case without matter (Fig 1(b)), while for $p = 1$ and $p = 2$ we have an additional term (Fig 1(c)). It is necessary to notice that the part of the separating curve beyond the Euclidian region is related to the so-called Euclidean counterparts of the equations of motion studied in quantum cosmology (see [8]) and will not be considered here.
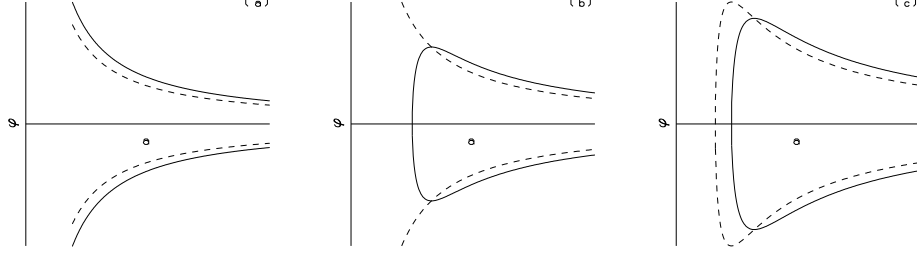
Figure 1: Configurations of the Euclidean boundary (solid) and the separating curve (dashed), in Fig. 1(a) without hydrodynamical matter, in Fig. 1(b) in the presence of matter with $p = 4$, in Fig. 1(c) in the presence of a fluid with $p \neq 4$.

In all cases the separating curve crosses the Euclidean boundary at

$$a_{\mathrm{cr}}^p = \frac{4\pi(2+p)}{3m_P^2}Q, \qquad (3.7)$$

Keeping in mind this feature and the geometry of the Euclidean region, it is sufficiently simple to explain the numerical results plotted in Fig. 2.

The influence of matter is significant only for small values of the scale factor which correspond to small values of z in Fig. 2. For Q small enough to keep $a_{\min}$ greater than a_{cr}, the situation is like that in Fig. 2(b): only trajectories with small initial values of z "feel" the presence of additional matter and can change their behaviour significantly. The measure of trajectories falling to singularity slowly increases with increasing Q. When $a_{\min} = a_{\mathrm{cr}}$, all the trajectories with zero velocity fall to the singularity because their initial points lie lower than the separating curve (see Fig. 2(c)). A further increase of Q leads to the situation of Fig. 2(d) - there exists some minimal value of z which is necessary for reaching the inflationary asymptotic. The measure of such trajectories keeps on diminishing with increasing Q.

But for

$$Q = (\frac{3}{8\pi})^{1+p/2}m_P^{2-p} \qquad (3.8)$$

$a_{\min}$ becomes equal to the value (3.5). The Euclidean boundary does not exist any more at $a_{\min}$. This situation corresponds to a density of matter so large, that a large spatial curvature (small z) is incompatible with the restriction of the initial density by the Planck value. The value $z_{\min}$ bounds the physically admissible initial conditions with densities smaller than the Planck density (Fig. 2(e)). In such a situation the fraction of initial conditions leading to inflation among all physically admissible initial conditions ($z > z_{\min}$) decreases with increasing Q (see Figs. 2(e) – 2(f)).

So the measure of non-inflating trajectories as a function of the initial density of ordinary matter has a maximum. This maximum corresponds to such Q that the density of ordinary matter at $z = 0$ has just the Planck
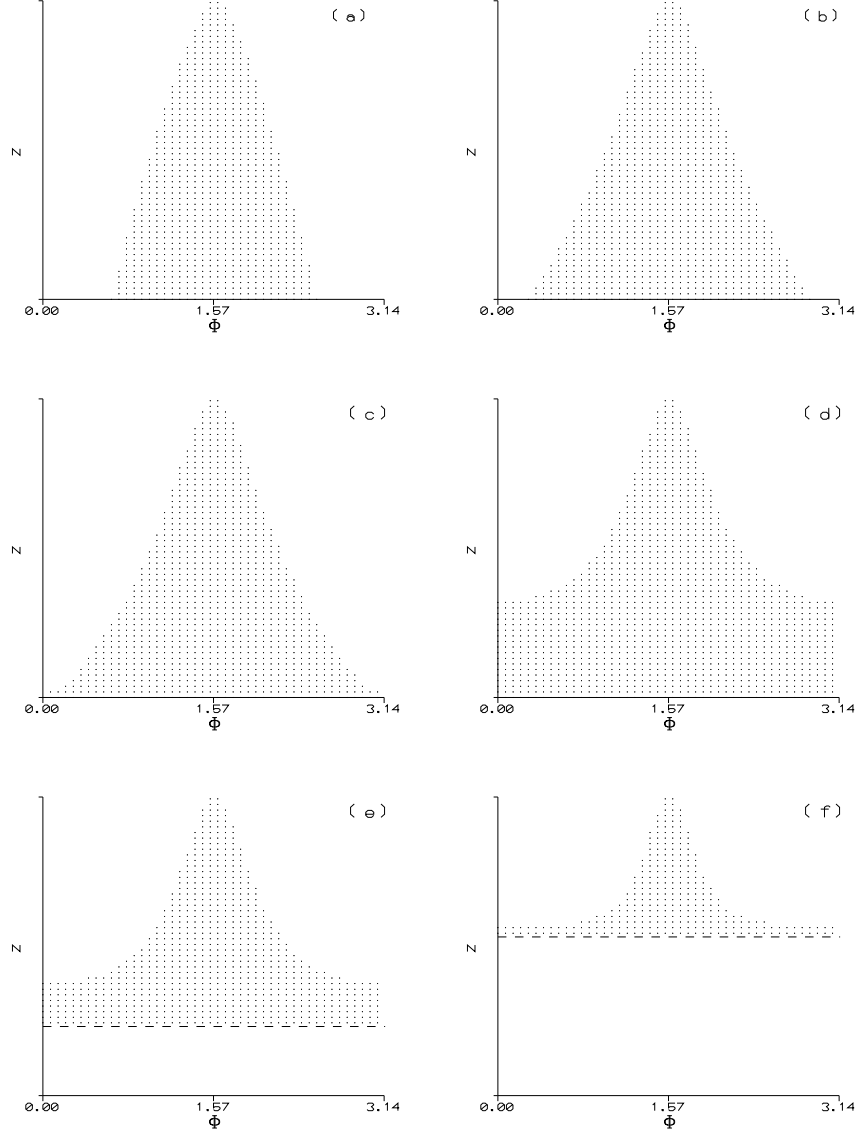
Figure 2: The area of initial data on the Planck boundary which do not lead to inflation. Fig. 2(a) corresponds to the situation without hydrodynamical matter, other plots correspond to densities of the hydrodynamical matter increasing from Fig. 2(b) to Fig. 2(f) (see details in the text). The value of ϕ varies is the range $0 \leq \phi \leq \pi$ and the value of z in the range $0 \leq z \leq z_{max}$.

7

value. The numerical value of this maximum slowly depends on p. For the three cases mentioned above the numerical values are

$$0.55 \text{ for } p = 1 \ (\gamma = 0),$$
$$0.56 \text{ for } p = 2 \ (\gamma = 1/3),$$
$$0.58 \text{ for } p = 4 \ (\gamma = 1).$$

4   Discussion

The results in [4, 5] are not essentially changed if we start not exactly from the Planck energy but from smaller values. This is important because we can not be sure that our model is valid up to exactly the Planck energy scale. Indeed, the essential criterion for inflation is that typical initial values of φ lie in the slow-rolling region. The value of φ_{sep} separating the slow-roll (for $\varphi > \varphi_{\text{sep}}$) and oscillatory (for $\varphi < \varphi_{\text{sep}}$) regimes can be estimated as $\varphi_{\text{sep}} = m_P/(2\sqrt{\pi})$.

Now, if we start from the energy density $E_{\text{in}} = \epsilon m_P^4$, $\epsilon < 1$, the maximum possible initial value of φ is $\varphi = \sqrt{2E_{\text{in}}}/m$ and the condition $\varphi \gg \varphi_{\text{sep}}$ leads to $m^2/m_P^2 \ll \epsilon$. If this condition is satisfied, than only a tiny part of the trajectories (with the measure $\epsilon^{-1/2} m/m_P$) falls into an oscillatory regime with an insufitient degree of inflation while the main part of non-inflationary trajectories falls into a singularity due to the spatial curvature and their measure is almost independent of the scalar field mass [4].

The presence of ordinary matter does not change the slow-roll regime, so all the aforesaid about the influence of the scalar field mass on the measure of non-inflationary trajectories is still valid.

The configuration of the Euclidean boundary and the separating curves in the presence of matter depends on the value of the scale factor. Value of the initial energy density deftermnes the minimal possible value of the scale factor (see the constraint equation) and therefore can influence the results. But it is clear from (3.5), (3.7) that the configuration of the curves is invariant under transformations keeping Q/a^p constant. Using the equation for a_{min}, this condition can be rewritten as $Q\epsilon^{p/2} = const$. Thus the transformations

$$E_{\text{in}} \rightarrow \epsilon E_{\text{in}},$$
$$Q \rightarrow \epsilon^{-p/2} Q$$

leave the situation unchanged. This symmetry of Euclidean and separating curves indicates that the maximum fraction of non-inflating trajectories does not depend on ϵ. The maximum fraction is achieved when the density of ordinary matter at a_{min} is equal to E_{in}. These qualitative considerations were also confirmed by direct numerical integration of the equations of motion.

As a result, the presence of a perfect fluid with $0 \leq \gamma \leq 1$ in the Universe filled by a massive scalar field can enlarge the fraction of non-inflationary trajectories, but this fraction cannot exceed $\sim 60\%$ and the inflationary asymptotic remains rather natural.

Acknowledgments

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